Controlling the orbital angular momentum of Gaussian vortices by shifting the point of phase singularity

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Abstract. A simple formula is obtained to describe the normalized orbital angular momentum (OAM) of a Gaussian beam after passing through a shifter spiral phase plate (SPP). The formula shows that while being equal to the topological charge at the zero off-axis shift, the OAM becomes fractional with increasing shift and it is tending to zero exponentially. Analytic expressions of the complex amplitude of the Gaussian beam having passed through the off-axis SPP show that as the beam propagates, the isolated intensity null moves from the initial point defined by the vector of the SPP's center shift along a straight line perpendicular to the said vector. Using a liquid crystal light modulator, crescent-shaped beams are experimentally generated.

1. Introduction

It is known shifting the centers of a Gaussian beam and a 'forked' amplitude hologram allows generation of asymmetric optical vortices with a fractional OAM [1, 2]. Such vortices allow to detect a photon pair with the entanglement of the OAM states. Transformation of classical optical vortices has also been reported, with the optical vortex understood as a laser beam characterized by an isolated on-axis intensity null and a spiral phase with integer topological charge. Thus, effects of the on-axis shift of the Gaussian beam waist from the plane of a spiral phase plate (SPP) combined with a diffractive (spiral) lens on the optical vortex have been theoretically studied [4]. Passage of an optical vortex through a set of pinholes centered on a circle in an opaque screen has been experimentally studied [5]. The degenerate optical vortex of the n-th order was shown to disintegrate into n non-degenerate optical vortices of the first order. Transformation of an optical vortex via introducing varying-degree ellipticity was also studied in [6], which was a follow-up of earlier research on related topics [7, 8]. Another approach to generating optical vortices [9] utilizes a set of pinholes arranged along a spiral in an opaque screen. Minor deviations of the guiding spiral were found to result in a distorted shape of the vortex.

Most closely related to the present topic are our earlier papers [10-12] in which we studied theoretically and experimentally optical vortex transformations due to a complex shift of the initial

complex amplitude function in the Cartesian coordinates. Such a shift led to asymmetry of the optical vortex, producing a crescent-shaped intensity pattern, rather than a ring-shaped or a doughnut intensity. The method of the complex shift of coordinates has been applied to a Bessel beam [10] and a Laguerre-Gaussian beam [11]. Conversion of an n-th order optical vortex into n optical vortices of the first order using an elliptic Gaussian beam incident on an SPP has also been described [12]. A relationship to describe the normalized OAM of a Gaussian beam implanted with an off-axis optical vortex (with n-times degenerate intensity null) has been deduced [13]. The OAM was shown to decrease quadratically with increasing distance from the Gaussian beam center and the isolated intensity null.

Unlike Ref. [13], in this work we analyze a more realistic situation of a Gaussian beam passing through an SPP with topological charge n, with the centers thereof being mutually shifted in the waist plane. A simple formula to describe the normalized OAM of the Gaussian beam having passed through the off-axis SPP with integer topological charge is deduced. This formula is different from a similar formula proposed in [13] and shows that at the zero-shift, the OAM equals n, becoming fractional and exponentially tending to zero as the shift decreases. The derived analytic expressions for the complex amplitude of the Gaussian beam having passed through the off-axis SPP suggest that as the beam propagates, the isolated intensity null moves from the initial point defined by the SPP center's shift vector along a straight line perpendicular to the SPP center's shift vector. Using a liquid crystal spatial light modulator, such beams with a crescent-shaped intensity pattern are experimentally generated. Thus, the OAM of the beams of interest is shown to be fractional, which means that the light field is composed of a linear combination of an enumerable set of optical vortices with integer topological charges. It is the simplest way to detect the entanglement of the OAM states of photons. Note that the complex shift of coordinates causes the OAM of Bessel [10] and Laguerre-Gaussian beams [11] to increase, the same being the case for an elliptic Hermite-Gaussian vortex beam [12]. On the contrary, the asymmetric Gaussian beam described in [13] and the Gaussian vortex with fractional OAM discussed in this work are characterized by the OAM decreasing with increasing spacing between the centers of the Gaussian beam and the SPP. Note, however, that the decrease follows different patterns: quadratic [13] and exponential (this study).

2. Orbital angular momentum of a Gaussian vortex after passing a shifted SPP

Let there be a Gaussian beam whose amplitude at the waist is given by

$$E_0\left(x, y, z = 0\right) = \exp\left(-\frac{x^2 + y^2}{w^2}\right), \tag{1}$$

where w is the waist radius. Assume that beam (1) falls on an SPP with number n, with its center being shifted with respect to the Gaussian beam's center by x_0. After passing the SPP, complex amplitude of the beam reads as

$$E_n\left(x, y, z = 0\right) = \frac{\left[\left(x - x_0\right) + i\left(y - y_0\right)\right]^n}{\left[\left(x - x_0\right)^2 + \left(y - y_0\right)^2\right]^{n/2}} \exp\left(-\frac{x^2 + y^2}{w^2}\right). \tag{2}$$

In polar coordinates, Eq. (2) is as follows:

$$E_n\left(x, y, z = 0\right) = \frac{\left(re^{i\varphi} - r_0 e^{i\varphi_0}\right)^n}{\left[r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)\right]^{n/2}} \exp\left(-\frac{r^2}{w^2}\right), \tag{3}$$

where $r^2 = x^2 + y^2$, $r_0^2 = x_0^2 + y_0^2$, $\tan\varphi = y/x$, $\tan\varphi_0 = y_0/x_0$.

It has been experimentally demonstrated (Mair et al. 2001; Vaziri et al. 2002; Chen et al. 2008) that the shift between the centers of a Gaussian beam and an optical vortex allows to detect the entanglement of the OAM states of photons. That is, after passing an off-axis SPP the beam is supposed to carry a fractional OAM. Below, we show this to be the case. Considering that the OAM of a propagating laser beam is preserved, it can be calculated in an arbitrary plane, e.g. in the waist plane. The energy W_n of the beam (3) and the z-projection of the OAM can be shown to be given by

$$W_n = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \left|E_n\left(x,y,z=0\right)\right|^2 dxdy = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \exp\left(-2\frac{x^2+y^2}{w^2}\right)dxdy = \frac{\pi w^2}{2} = W_0. \tag{4}$$

$$J_{nz} = \mathrm{Im}\left\{\int\limits_0^\infty\int\limits_0^{2\pi} \overline{E}_n \frac{\partial E_n}{\partial\varphi} rdrd\varphi\right\}. \tag{5}$$

Substituting Eq. (3) into Eq. (5), we obtain

$$J_{nz} = n\frac{\pi w^2}{2}\exp\left(-\frac{2r_0^2}{w^2}\right). \tag{6}$$

Thus, the normalized OAM is described by a simple formula:

$$\frac{J_{nz}}{W_n} = n\exp\left(-\frac{2r_0^2}{w^2}\right). \tag{7}$$

From (7), the normalized OAM is seen to depend only on the radius r_0 of the off-axis shift from the SPP center, being independent of the shift angle φ_0. At $r_0 = 0$, Eq. (7) can be rearranged, describing a well-known statement that the OAM of the Laguerre-Gaussian mode is equal to the topological charge n (Allen et al. 1992):

$$\frac{J_{nz}}{W_n} = n. \tag{8}$$

Equation (7) also suggests that at a large shift $r_0 \rightarrow \infty$ the OAM tends to zero. Physically, in this case the SPP's center is found outside the Gaussian beam.

3. Propagation dynamics of a Gaussian beam after passing a shifted SPP
At distance z from the SPP, the amplitude of the light field is expressed via a Fresnel transform as

$$E_n\left(x',y',z\right) = \frac{-ik}{2\pi z}\exp\left(ik\frac{x'^2+y'^2}{2z}\right)\times$$
$$\times\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} E_n\left(x,y,z=0\right)\exp\left(ik\frac{x^2+y^2}{2z}\right)\exp\left(-ik\frac{xx'+yy'}{z}\right)dxdy. \tag{9}$$

Changing to polar coordinates, the double integral in (9) can be calculated via a reference integral (Prudnikov et al., 1981):

$$\int\limits_0^\infty e^{-pr^2} J_n(cr)rdr = \frac{\sqrt{\pi}t}{2\sqrt{2}p}\exp(-t)\left[I_{(n-1)/2}(t) - I_{(n+1)/2}(t)\right], \tag{10}$$

where $t = c^2/(8p)$ and $J_n(x)$, $I_n(x)$ are a Bessel function and a modified n-th order Bessel function.

In this section, for simplicity, we suppose that $y_0 = 0$. Then, Eq. (9) reduces to

$$E_n\left(x',y',z\right) = \frac{\sqrt{\pi}}{\sqrt{2}q(z)}\left(\frac{iA-B}{\sqrt{A^2+B^2}}\right)^n \exp\left[-\left(\frac{x_0}{w}\right)^2 + \frac{ikx_0^2}{2z} + ik\frac{x'^2+y'^2}{2z} - \frac{ikx_0x'}{z}\right]$$
$$\times\sqrt{t}\exp(-t)\left[I_{(n-1)/2}(t) - I_{(n+1)/2}(t)\right], \tag{11}$$

where

$$A = -\frac{kx'}{z} + \frac{kx_0}{z} + \frac{2ix_0}{w^2}, \quad B = -\frac{ky'}{z}, \quad q(z) = 1 + \frac{iz}{z_0},$$

$$t = \frac{\mathrm{Re}+i\,\mathrm{Im}}{2w^2\left|q(z)\right|^2},$$

$$\mathrm{Re} = x'^2 + y'^2 - \left(x_0\left|q(z)\right|\right)^2, \tag{12}$$

$$\mathrm{Im} = \left[x'^2 + y'^2 + x_0^2\left|q(z)\right|^2 - 2x_0x'\left|q(z)\right|^2\right]\left(\frac{z_0}{z}\right).$$

From Eq. (11), an equation can be obtained for the trajectory of the isolated intensity null (singularity point). Making the real and imaginary parts of the argument t in Eq. (12) equal to zero, we get:

$$\begin{cases} x'^2 + y'^2 - x_0^2 |q(z)|^2 = 0, \\ x'^2 + y'^2 + x_0^2 |q(z)|^2 - 2x_0 x' |q(z)|^2 = 0. \end{cases} \tag{13}$$

The solution of the system (13) reads as

$$\begin{cases} x' = x_0, \\ y' = x_0 \dfrac{z}{z_0}. \end{cases} \tag{14}$$

From Eq. (14) it follows that the isolated intensity null of the vortex beam (11) shifts along a vertical line orthogonal to the axis x' and this shift is linearly proportional to the propagation distance z from the initial plane. Since the Gaussian beam is rotationally symmetric, shifting the SPP center by the vector (x_0, y_0) leads to the orthogonal shift of the isolated intensity null, which has the coordinates $(x_0 - y_0 z / z_0, \ y_0 + x_0 z / z_0)$.

### 4. Experimental generation of Gaussian beams with fractional OAM

Figure 1 shows an experimental setup. The light from a solid-state laser of wavelength $\lambda = 532$ nm was passed through a pinhole PH and lens L_1 before hitting the SLM's display, on which a phase function of a given-order SPP had been output. The beam reflected at the SLM was spatially filtered using a pair of lenses L_2 and L_3 and a diaphragm D. After filtering, the laser beam was directed to the lens L_4, which focused it on the CMOS-camera. To be able to obtain interferograms, the optical setup had been complemented with beam splitting cubes BS_1 and BS_2. The cube BS_1 divided the original beam in two beams, with one of them being directed toward the SLM and the other remaining unchanged. Using the beam splitting cube BS_2, the two beams were then reunited, making it possible to observe the resulting interference pattern on the camera display. The role of lens L_5 was to introduce a spherical wavefront into the Gaussian beam. The Gaussian beam's waist diameter was $2w = 1400$ µm.

In the experiments, we studied effects caused by the shift between the centers of the illuminating beam and the SPP output on the SLM display. Figure 2 depicts intensity patterns generated in the focal plane of a lens with focal length $f = 250$ mm. The figures also depict corresponding interferograms resulting from the interference of the beams under study and a Gaussian beam with spherical wavefront. In Fig. 2, each picture is 750×750 µm in size. It is worth noting that the beam splitting cube in the path of the laser beam reflected from the modulator causes the focal plane of lens L_2 to be shifted toward the SLM. As a result, the laser beam incident on lens L_4 was converging and the Fraunhofer diffraction pattern was observed at distance $z = 230$ mm from the plane of lens L_4.
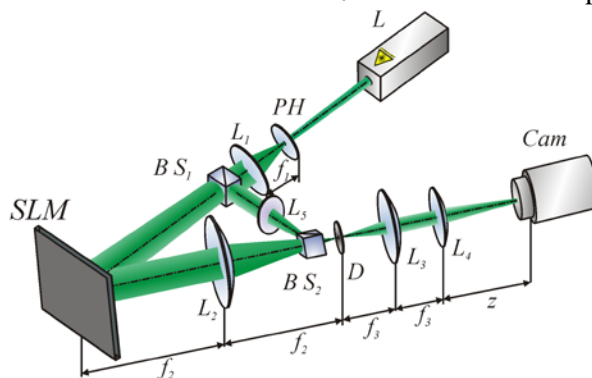
Figure 1. Experimental setup: L is a solid-state laser ($\lambda = 532$ nm), PH – pinhole (40 µm), L_1, L_2, L_3, L_4 and L_5 are lenses with focal lengths $f_1 = 150$ mm, $f_2 = 350$ mm, $f_3 = 150$ mm, $f_4 = 250$ mm, $f_5 = 150$ mm), BS_1, BS_2 are beam splitting cubes, SLM is a spatial light modulator (SLM) PLUTO VIS (1920×1080 resolution, 8-µm pixels size), D is a diaphragm to perform spatial filtering, Cam is a 1.67-µm pixel video-camera ToupTek U3CMOS08500KPA.

Figure 2 shows intensity patterns for the Gaussian optical vortices with the topological charge n ranging from 1 to 4 given a significant shift $x_0 = 0.5w$ between the centers of the SPP and the Gaussian beam. Figure 2 suggests that, first, the intensity patterns have changed from doughnut to crescent-shaped and, second, the minimal intensity of the inhomogeneous intensity pattern is found on a line that makes an angle of 90° with the horizontal axis.
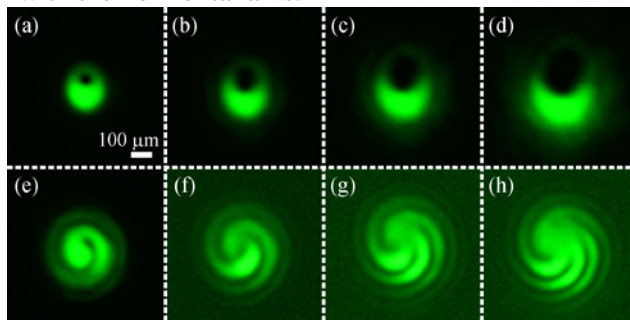
Figure 2. The shift between the centers of the SPP and the Gaussian beam is $x_0 = 0.5w$. Top row: intensity distributions for the SPP with the topological charge (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, and (d) $n = 4$. Bottom row: corresponding interferograms for the SPP with the topological charge (e) $n = 1$, (f) $n = 2$, (g) $n = 3$, and (h) $n = 4$. The pictures are 750×750 µm in size.

5. Conclusions

Thus, we have derived a simple formula to describe the normalized OAM of a Gaussian beam having passed through an off-axis SPP with integer topological charge n. The derived relationship shows that while being equal to n at the zero off-axis shift, the OAM becomes fractional with increasing shift, exponentially tending to zero. Analytic relationships for the complex amplitude of a Gaussian beam having passed through an off-axis SPP have been deduced, showing that as the beam propagates, the isolated intensity null moves from the initial point defined by the SPP shift vector along a straight line perpendicular to the shift vector. The beams of interest with a crescent-shaped transverse intensity pattern have been generated experimentally using a liquid crystal light modulator. The decrease of the OAM with increasing inter-center shift can be explained by the fact that the optical vortex turns out to be in a low-energy region.

6. Acknowledgements

7. References

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