Reconstruction of vector (signal) by the norms of projections

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Abstract

The foundations of the frame theory in finite-dimensional Euclidean space are represented. The ability of frames in the reconstruction of the vector signal without phase measurements are shown. There is a review of a number of new concepts and their role in the signal reconstruction. The possibility of reconstruction of the vector by the norms of the projections on the subspaces is asserted. Particular attention is paid to systems of subspaces for which there is the possibility of reconstruction by the norms of the projections on them and on their orthogonal complements.

Keywords: frame; phaseless reconstruction; complement property; full spark set; norm retrieval

1. Basic facts. Phaseless recovery

Let $\mathbb{H}^M$ denotes M-dimensional space with the scalar product.

Definition 1. A set of vectors $\Phi = \{\varphi_k\}_{k=1}^N$ is called a *frame* for the $\mathbb{H}^M$, if there are positive constants A, B such that for all $x \in \mathbb{H}^M$

$$A\|x\|^2 \leq \sum_{k=1}^{N} |\langle x, \varphi_k\rangle|^2 \leq B\|x\|^2.$$

Numbers A and B are called the lower and upper frame bounds respectively. If we can choose $A = B$, then the frame is called *tight*, and if $A = B = 1$, it is called a *Parseval-Steklov frame* (This name was proposed by Acad. V.S. Vladimirov during a report of the second author in Math. Steklov Institute in 2008 instead of usual Parseval Frame).

Note that in the finite dimensional setting, a frame is simply a spanning set of vectors in the Hilbert space ($\text{span}\{\varphi_k\}_{k=1}^N = \mathbb{H}^M$) [1, 2].

There are three operators connected with a frame Φ:
analysis operator $T : \mathbb{H}^M \to \ell_2^N$, defined by

$$T(x) = \{\langle x, \varphi_k\rangle\}_{k=1}^N,$$

adjoint *synthesis operator*

$$T^*\left(\{a_k\}_{k=1}^N\right) = \sum_{k=1}^{N} a_k\varphi_k$$

and *frame operator* $S := T^*T$ on $\mathbb{H}^M$, defined by

$$S(x) = T^*T(x) = \sum_{k=1}^{N}\langle x, \varphi_k\rangle\varphi_k.$$

The frame operator is positive, self-adjoint and invertible. Besides, we have

$$AI \leq S \leq BI,$$

where I is identity operator in $\mathbb{H}^M$.

In particular, for the Parseval-Steklov frame the frame operator is the identity operator, so this frame is the most useful for the reconstruction of signals. In fact, in this case for every $x \in \mathbb{H}^M$ the following equality is true

$$x = \sum_{k=1}^{N}\langle x, \varphi_k\rangle\varphi_k.$$

The operator $G = TT^*$ is Gram operator with the matrix

$$\begin{pmatrix} \|\varphi_1\|^2 & \langle\varphi_2, \varphi_1\rangle, & \dots & \langle\varphi_N, \varphi_1\rangle \\ \langle\varphi_1, \varphi_2\rangle & \|\varphi_2\|^2 & \dots & \langle\varphi_N, \varphi_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varphi_1, \varphi_N\rangle & \langle\varphi_2, \varphi_N\rangle & \dots & \|\varphi_N\|^2 \end{pmatrix}$$

and for the Parseval-Steklov frame coincides with the projector $P : \ell_N^2 \to \ell_N^2$ to the image of the analysis operator [1, 2].

An easy way is khown to construct Parseval-Steklov frames. It is based on the following proposition.

Proposition 1. *Let $\{\varphi_k\}_{k=1}^N$ be a frame for $\mathbb{H}^M$ with bounds A and B, and let P be the orthogonal projection in $\mathbb{H}^M$ on the subspace W. Then $\{P\varphi_k\}_{k=1}^N$ is a frame for W with bounds A and B. In particular, if $\{\varphi_k\}_{k=1}^N$ is Parseval-Steklov frame for $\mathbb{H}^M$ and P is the orthogonal projection on W, then $\{P\varphi_k\}_{k=1}^N$ is Parseval-Steklov frame for W.*

Proof. We have for $x \in W$

$$A\|x\|^2 = A\,\|Px\|^2 \leq \sum_{k=1}^{N} |\langle Px, \varphi_k\rangle|^2 =$$

$$= \sum_{k=1}^{N} |\langle x, P\varphi_k\rangle|^2 \leq B\|Px\|^2 = B\|x\|^2.$$

Corollary 1. *Let $\{e_k\}_{k=1}^M$ be an orthonormal basis (ONB) in $\mathbb{H}^M$, and let P be the orthogonal projection on the subspace W. Then $\{Pe_k\}_{k=1}^M$ is Parseval-Steklov frame for W.*

Corollary 1 is the foundation of the following algorithm for construction of Parseval-Steklov frame. We construct $N\times N$ unitary matrix for $N \geq M$, then we choose any M rows, columns of thus obtaining $M \times N$-matrix form Parseval-Steklov frame in $\mathbb{H}^M$. If we construct from the remaining $N - M$ rows $(N - M) \times N$-matrix, then its columns are Parseval-Steklov frame in $\mathbb{H}^{N-M}$.

The following theorem, actually proved by Naimark, shows that such process is essentially the only one for constructing Parseval-Steklov frame [3].

Theorem 1.

Let $\Phi = \{\varphi_k\}_{k=1}^N$ be a frame in $\mathbb{H}^M$ with the analysis operator T, let $\{e_k\}_{k=1}^N$ be the standard basis in ℓ_N^2, let $P : \ell_N^2 \to \ell_N^2$ be the orthogonal projection on $\mathrm{Im}(T)$.

The following assertions are equivalent:

1. *Φ is Parseval-Steklov frame for $\mathbb{H}^M$.*
2. *For all $k = 1, \ldots, N$ we have $Pe_k = T\varphi_k$.*
3. *There are vectors $\{\psi_k\}_{k=1}^N \subset \mathbb{H}^{N-M}$ such that $\{\varphi_k \oplus \psi_k\}_{k=1}^N$ form ONB in $\mathbb{H}^N$.*

Besides, $\{\psi_k\}_{k=1}^N$ are Parseval-Steklov frame in $\mathbb{H}^{N-M}$.

Proof.

(1) $\Leftrightarrow$ (2). As noted, the system $\{\varphi_k\}_{k=1}^N$ forms Parseval-Steklov frame iff Gram operator TT^* coincides with the projection P. So (1) and (2) are equivalent according to equality $T^*e_k = \varphi_k$ for $k = 1, \ldots, N$.

(1) $\Rightarrow$ (3). Let's put $d_k = e_k - T\varphi_k$, $k = 1, \ldots, N$. According to (2), $d_k \in (\mathrm{Im}(T))^\perp$ for all k. For a unitary operator

$$\Phi : (\mathrm{Im}(T))^\perp \to \mathbb{H}^{N-M}$$

let's put

$$\psi_k := \Phi\, d_k,\ k = 1, \ldots, N.$$

We have using the isometry of the operator T,

$$\langle \varphi_i \oplus \psi_i, \varphi_k \oplus \psi_k\rangle = \langle \varphi_i, \varphi_k\rangle + \langle \psi_i, \psi_k\rangle =$$

$$= \langle T\varphi_i, T\varphi_k\rangle + \langle d_i, d_k\rangle = \delta_{ik}.$$

(3) $\Rightarrow$ (1). Let's apply corollary 1.

As in [4], we call vectors $\{\psi_k\}_{k=1}^N$ *Naimark complement of the frame* Φ .

For Parseval-Steklov frame, written as $\{Pe_k\}_{k=1}^N$, Naimark complement is the system of vectors $\{(I - P)e_k\}_{k=1}^N$.

Naimark complements are defined only for Parseval-Steklov frames, and they are defined up to unitary equivalence. If $\{\varphi_k\}_{k=1}^N \subset \mathbb{H}_M$ and $\{\psi_k\}_{k=1}^N \subset \mathbb{H}^{N-M}$ complement each other, U and V are unitary operators $(U^*U = UU^* = I)$, then $\{U\varphi_k\}_{k=1}^N$, $\{V\psi_k\}_{k=1}^N$ also complement each other.

An important application of frames is the reconstruction of a signal with incomplete data. In particular, much attention is attracted to the problem of the reconstruction phase information. In recent papers on this topic two aspects of the problem were emphasized: phaseless reconstruction and phase retrieval [7]. This paper focuses on the first aspect.

Definition 2. The set of vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^M$ (or $\mathbb{C}^M$) provides *phaseless reconstruction (PLR)*, if equalities of measurement modules

$$|\langle x, \varphi_i\rangle| = |\langle y, \varphi_i\rangle|, \quad x, y \in \mathbb{R}^M\ (\mathbb{C}^M), \quad i = 1, \ldots, N,$$

imply the equality of vectors-signals up to unimodular factor, i.e. $x = cy$ with some $c = \pm 1$ for $\mathbb{R}^M$ or $c \in \mathbf{T}$ for $\mathbb{C}^M$, where $\mathbf{T}$ is the unit circle in $\mathbb{C}$.

In the rest of the text sets, which are satisfied the definition of 2, is called PLR-systems or PLR-sets. The next property is important in these questions.

Definition 3 [4, 5]. The set $\Phi = \{\varphi_n\}_{n=1}^N$ in $\mathbb{H}^M$ has *complement property (CP)*, if for any $S \subseteq \{1, \dots, N\}$ $\{\varphi_n\}_{n\in S}$ or $\{\varphi_n\}_{n\in S^c}$ is complete in $\mathbb{H}^M$. Complement property in $\mathbb{R}^M$ is equivalent to PLR (theorem 2 below).

Definition 4 [4, 5, 6]. The *spark* of the set $\Phi = \{\varphi_n\}_{n=1}^N \subset \mathbb{H}^M$ is the cardinality of the smallest linear dependent subset of Φ. If $\text{spark}(\Phi) = M + 1$, then any subset with M vectors linear independent, in thus case Φ is called full spark set.

In earlier works the term "girth" was used instead of the term "spark". Spark of the linear independent system, for example, basic, is assumed to be zero.

Theorem 2 [5, 8].
Frame $\{\varphi_n\}_{n=1}^N$ in $\mathbb{R}^M$ is the PLR-system iff it has complement property. In particular, full spark frame with at least $2M - 1$ vectors is PLR-system. If $\{\varphi_n\}_{n=1}^N$ is PLR-system in $\mathbb{R}^M$, then $N \geq 2M - 1$, any subset with $2M - 2$ vectors can't be PLR-system.

Generally speaking, the recovery without phases is possible not only by full spark frames. Each frame, containing $(2M - 1)$ full spark frame, will also provide recovery without phases. However, if the frame contains exactly $2M - 1$ elements, it is a PLR-system only for full spark frame [5, 8].

Let's see if the possibility of recovery without phases to the is transferred to Naimark complements. We require the following theorem for this.

Theorem 3 [9].
Let P be an projection in $\mathbb{H}^N$ with ONB $\{e_n\}_{n=1}^N$ and $S \subset \{1, 2, \dots, N\}$.
The following assertions are equivalent:

1. $\{Pe_i\}_{i\in S}$ *linear independent.*
2. $\text{span}\,\{(I - P)e_i\}_{i\in S^c} = (I - P)\left(\mathbb{H}^N\right)$.

Proof.
(1) $\Rightarrow$ (2). Let's suppose, that
$$\text{span}\,\{(I - P)e_i\}_{i\in S^c} \neq (I - P)\left(\mathbb{H}^N\right).$$
It means, that there exists $0 \neq x \in (I - P)\left(\mathbb{H}^N\right)$ such that $x \perp \text{span}\,\{(I - P)e_i\}_{i\in S^c}$. As $x = \sum_{i=1}^{N} \langle x, e_i\rangle (I - P)e_i$, then
$$\langle x, (I - P)e_i\rangle = \langle (I - P)x, e_i\rangle = \langle x, e_i\rangle = 0$$
for any $i \in S^c$. Hence, $x = \sum_{i\in S} \langle x, e_i\rangle e_i$, so
$$\sum_{i\in S} \langle x, e_i\rangle e_i = x = (I - P)x = \sum_{i\in S} \langle x, e_i\rangle (I - P)e_i,$$
i.e. $\sum_{i\in S} \langle x, e_i\rangle Pe_i = 0$, and, thus, $\{Pe_i\}_{i\in S}$ are linearly dependent.

(2) $\Rightarrow$ (1). Let's suppose, that $\{Pe_i\}_{i\in S}$ are linearly dependent: there exist numbers $\{b_i\}_{i\in S}$, among which there are nonzero, and $\sum_{i\in S} b_i Pe_i = 0$. Then
$$x := \sum_{i\in S} b_i (I - P)e_i = \sum_{i\in S} b_i e_i \in (I - P)\left(\mathbb{H}^N\right).$$
Let's consider
$$\langle x, (I - P)e_j\rangle = \langle (I - P)x, e_j\rangle = \left\langle \sum_{i\in S} b_i e_i, e_j \right\rangle = \sum_{i\in S} b_i \langle e_i, e_j\rangle = 0,$$
if $j \in S^c$. Thus, $x \perp \text{span}\,\{(I - P)e_i\}_{i\in S^c}$, and hence,
$$\text{span}\,\{(I - P)e_i\}_{i\in S^c} \neq (I - P)\left(\mathbb{H}^N\right).$$

Proposition 2. *Parseval-Steklov frame is a full spark frame iff Naimark complement of this frame is a full spark frame also.*
Proof. By theorem 1, Parseval-Steklov frame can be written as $\{Pe_i\}_{i=1}^N$, where $\{e_i\}_{i=1}^N$ is an ONB in $\mathbb{H}^N$ and P is the orthogonal projection in $\mathbb{H}^N$. Naimark complement for Parseval-Steklov frame looks as $\{(I - P)e_i\}_{i=1}^N$. By definition $\{Pe_i\}_{i=1}^N$ is a full spark frame, if for any $S \subseteq \{1, \dots, N\}$ with $|S| = M$ $\{Pe_i\}_{i\in S}$ is a basis in the range of the projection P. By theorem 3, we have that $\{(I - P)e_i\}_{i\in S^c}$ is a basis in the range of the projection $I - P$, so $\{(I - P)e_i\}_{i=1}^N$ is a full spark frame also. The reverse assertion is proved similarly.

If Parseval-Steklov frame ensures recovery without phases, Naimark complement can not provide recovery without phases. The thing is including, in particular, that in Naimark complement may be insufficient number of vectors.

Proposition 3. *If Parseval-Steklov frame $\{\varphi_n\}_{n=1}^N$ ensures recovery without phases in $\mathbb{R}^M$, and Naimark complement to this frame also ensures recovery without phases in $\mathbb{R}^{N-M}$, then*
$$2M - 1 \leq N \leq 2M + 1.$$

Proof. If $\{\varphi_n\}_{n=1}^N$ ensures recovery without phases in $\mathbb{R}^M$, then $N \geq 2M-1$ (theorem 2). If Naimark complement ensures recovery without phases in $\mathbb{R}^{N-M}$, then $N \geq 2(N-M)-1$, or $N \leq 2M+1$.

But Naimark complement can fail to ensure recovery without phases even under conditions of proposition 3.

Example. Let $\{\varphi_m\}_{m=2}^{2M}$ be the full spark frame in $\mathbb{R}^M$, $M \geq 3$. Let's put $\varphi_1 = \varphi_2$, and let S be the frame operator for $\{\varphi_m\}_{m=1}^{2M}$. Note that $\left\{S^{-\frac{1}{2}}\varphi_m\right\}_{m=2}^{2M}$ is full spark frame, and ensures recovery without phases. For any partition $\mathcal{S}, \mathcal{S}^c \subset \{1,\ldots,2M\}$ one of the sets $\mathcal{S}$ or $\mathcal{S}^c$ has at least M elements from the full spark frame $\left\{S^{-\frac{1}{2}}\varphi_m\right\}_{m=2}^{2M}$ and hence complete in $\mathbb{R}^M$.

Now let's show, that Naimark complement for $\left\{S^{-\frac{1}{2}}\varphi_m\right\}_{m=1}^{2M}$ does not ensure recovery without phases. Let's break $\left\{S^{-\frac{1}{2}}\varphi_m\right\}_{m=1}^{2M}$ on $\left\{S^{-\frac{1}{2}}\varphi_m\right\}_{m=1}^{2}$ and $\left\{S^{-\frac{1}{2}}\varphi_m\right\}_{m=3}^{2M}$. None of them is linear independent, as $\varphi_1 = \varphi_2$, and $M \geq 3$. According to theorem 3, Naimark complements for each of these sets are not comlete in $\mathbb{R}^{2M-M} = \mathbb{R}^M$. Thus, there is a partition of Naimark complement which contradicts the complement property and does not ensure phaseless recovery.

If Parseval-Steklov frame is full spark frame, then phaseless recovery is inherited by Naimark complement.

Proposition 4. *If $\Phi = \{\varphi_n\}_{n=1}^N$ is full spark Parseval-Steklov frame, $2M-1 \leq N \leq 2M+1$, then Φ ensures phaseless recovery in $\mathbb{R}^M$, and Naimark complement for Φ ensures phaseless recovery in $\mathbb{R}^{N-M}$.*

Proof. By proposition 2 Naimark complement for Φ is full spark frame in $\mathbb{R}^{N-M}$. We have $2M-1 \leq N$ and $2(N-M)-1 \leq N$, then, by theorem 2, both Φ and its Naimark complement have complement property in relevant spaces.

2. Recovery by the norms of projections

Following [4, 10] we define the recovery of a vector-signal by the norms of projections on subspaces.

Definition 5. Let $\{W_n\}_{n=1}^N$ be the set of subspaces in $\mathbb{H}^M$, let $\{P_n\}_{n=1}^N$ be orthogonal projections on these subspaces.
We say, that $\{W_n\}_{n=1}^N$ *(or $\{P_n\}_{n=1}^N$) ensures recovery by the norms of projections*, if for any $x, y \in \mathbb{H}^M$ equalities $\|P_n x\| = \|P_n y\|$ for $n = 1,\ldots,N$ imply $x = cy$ for some c with $|c| = 1$.
Further such sets of subspaces will be called *RNP-sets*.

A lot of attention to such recovery is paid in [10]. For one-dimensional subspace W_n the number $\|P_n x\|$ can be received only from two vectors $\pm P_n x$. For subspaces W_n with higher dimensions we have continuum of vectors with $\|P_n x\|$.
Nevertheless the map

$$\mathcal{A}(x)(n) = \|P_n x\|$$

can be injective for subspaces with higher dimensions. The proof of this result uses the scheme of [10], we need some auxiliary assertions.

Lemma 1.
Let $\{\varphi_n\}_{n=1}^N$ be full spark frame in $\mathbb{R}^M$. Let's define ONB in $\mathbb{R}^M$ using the following algorithm: ψ_1 is a random vector, ψ_2 is a random vector from $[\mathrm{span}(\psi_1)]^\perp, \ldots, \psi_k$ is a random vector from $\left[\mathrm{span}(\{\psi_n\}_{n=1}^{k-1})\right]^\perp$. Then $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^M$ is the full spark frame with the probability 1.

Proof.
Let $1 \leq k < M$. We suppose, that $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^k$ is full spark frame, we need to check, that $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^{k+1}$ is full spark frame too. For this we have to show that ψ_{k+1} does not lie in the span of any $M-1$ vectors from $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^k$. Choose any $M-1$ such vectors and denote them by A. Put $W_k := \left[\mathrm{span}(\{\psi_m\}_{m=1}^k)\right]^\perp$ and pick ψ_{k+1} as a random unit norm vector from this $(M-k)$ -dimensional space. Then $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^{k+1}$ is full spark system $\Leftrightarrow \psi_{k+1} \notin \mathrm{span}(A)$. The last is truly with probability 1 iff

$$\dim(\mathrm{span}(A) \cap W_k) \leq (M-k)-1. \tag{1}$$

In fact, $\mathrm{span}(A) \cap W_k$ is a subset in $(M-k)$ -dimensional space W_k, and so inequality (1) implies that this intersection has zero measure. Hence, we have with probability 1 $\psi_{k+1} \notin \mathrm{span}(A) \cap W_k$ and $\psi_{k+1} \in W_k$. Now we are going to the proof of inequality (1).
Let's apply the method of mathematical induction. A vector ψ_1 is chosen randomly from $W_0 = \mathbb{R}^M$. If A any $M-1$ vectors from $\{\varphi_n\}_{n=1}^N$, then

$$\dim(\mathrm{span}(A) \cap W_0) = M-1,$$

and $\{\varphi_n\}_{n=1}^N \cup \psi_1$ is full spark frame with probability 1.
Let's suppose that $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^k$ is full spark frame. We denote by A any $M-1$ vectors from $\{\varphi_n\}_{n=1}^N \cup \{\psi_m\}_{m=1}^k$.
Let's consider two possible cases.
1. $\psi_k \notin A$. We have $W_k \subset W_{k-1}$ and

$$\mathrm{span}(A) \cap W_k = (\mathrm{span}(A) \cap W_{k-1}) \cap W_k.$$

Note that $\dim W_k = M-k$, $\dim(\mathrm{span}(A) \cap W_{k-1}) \leq M-k$, because $\psi_k \notin A$. So for the proof (1) it's suffice to check that these subspaces do not match. Let's suppose that $\mathrm{span}(A) \cap W_{k-1} = W_k$. We remember that $\psi_k \in W_k^\perp$, and hence, $\psi_k \in [\mathrm{span}(A) \cap W_{k-1}]^\perp$, this subspace has dimension k. As $\psi_k \notin W_{k-1}^\perp$, $\dim W_{k-1} = k-1$, and

$$W_k^\perp \subset [\mathrm{span}(A) \cap W_{k-1}]^\perp,$$

it turns to be that ψ_k lies in one-dimensional subspace, determined by span(A) and W_{k-1}. It's possible only with zero probability for randomly chosen vector from $M-(k-1)$-dimensional subspace W_{k-1}.

2. $\psi_k \in A$. Let's note that

$$\dim(\text{span}(A) \cap W_k) \le M-k,$$

as $\dim(W_k) = M-k$. For contradiction, we suppose that

$$\dim(\text{span}(A) \cap W_k) = M-k. \tag{2}$$

We have further that

$$W_k \subset \text{span}(A). \tag{3}$$

Pick $\varphi \in \{\varphi_n\}_{n=1}^N$ so that $\varphi \notin A$. Then

$$\dim(\text{span}(A \setminus \psi_k) \cap W_k) \le \dim(\text{span}(A \setminus \psi_k \cup \varphi) \cap W_k) \le (M-k)-1.$$

The last inequality is a result of the first case above.

On the other hand as $\psi_k \perp W_k$ and $\psi_k \in A$, we receive from (2) and (3)

$$\dim(\text{span}(A \setminus \psi_k) \cap W_k) = \dim(\text{span}(A) \cap W_k) = M-k.$$

This contradiction proves (1).

Corollary 2. *The finite set of ONB, which are built by the algorithm of random choice of lemma 1, if full spark frame with the probability 1.*

Proof. Let's apply consistently the lemma 1.

Lemma 2. *For an integer $M \ge 2$ let's pick integers $M-1 \ge I_1 \ge I_2 \ge \ldots \ge I_1 \ge 1$. There is a real invertible $M \times M$-matrix with $0-1$ instances such that the k-row has exactly I_k ones.*

Proof.

We apply induction by M. The claim is obvious for $M = 2$. Let's suppose that the assertion is valid for M. Let's look at the set of $M+1$ numbers such that

$$M = I_1 = \ldots = I_s > I_{s+1} \ge \ldots \ge I_{M+1} \ge 1$$

for some $s \le M+1$. By induction assumption for the set of numbers

$$I_1 - 1 = \ldots = I_s - 1 \ge I_{s+1} \ge \ldots \ge I_M \ge 1$$

there is the invertible $M \times M$-matrix $A = \left[a_{ij}\right]_{i,j=1}^M$ with $I_{k-1} - 1 = M-1$ ones in k-row for $k = 1, \ldots, s$ and I_k ones in k-row for $k = s+1, \ldots, M$. Let's define $(M+1) \times (M+1)$-matrix $B = \left[b_{ij}\right]_{i,j=1}^{M+1}$ defining

$$b_{ij} = \begin{cases} a_{ij}, & 1 \le i, j \le M, \\ 1, & 1 \le i \le s,\ j = M+1, \\ 1, & i = M+1,\ 1 \le j \le M+1, \\ 0, & \text{for other indexes.} \end{cases}$$

The matrix B has I_k ones in k-row for $k = 1, \ldots, M+1$. The matrix $A = \left[a_{ij}\right]_{i,j=1}^M = \left[b_{ij}\right]_{i,j=1}^M$ is invertible, so the matrix B by row reduces can be reduced to the step form $\widetilde{B} = \left[\widetilde{b}_{ij}\right]_{i,j=1}^{M+1}$, where $\left[\widetilde{b}_{ij}\right]_{i,j=1}^M = I_{M\times M}$, and the row $(M+1)$ is not changed. If we suppose that $\widetilde{B}$ is not invertible, then the row $(M+1)$ by row reduces can be reduced to the zero row and hence

$$\sum_{i=1}^{I_{M+1}} \widetilde{b}_{M+1,i} = 0. \tag{5}$$

Let's define for each $l \in \{1, \ldots, I_{M+1}\}$ the matrix $\widetilde{B}_l$. It is obtained from the matrix $\widetilde{B}$ changing $\widetilde{b}_{M+1,M+1} = 0$ to $\widetilde{b}_{M+1,l} = 1$.

If $\widetilde{B}$ is not invertible, then by row reduces the last row is reduced to the zero row, and we have

$$\sum_{i=1, i\neq l}^{I_{M+1}} \widetilde{b}_{M+1,i} = -1. \tag{6}$$

The equality (6) is valid for any $l \in \{1, \ldots, I_{M+1}\}$, that's contradict to (5). Hence at least one of the matrixes $\widetilde{B}$ or $\widetilde{B}_l$ for some $l \in \{1, \ldots, I_{M+1}\}$ has to be invertible.

Theorem 4. *There exists RNP-set in $\mathbb{R}^M$ consisting from $2M-1$ subspaces, dimension of each subspace $< M-1$.*

Proof.

Let $\{\varphi_n\}_{n=1}^{2M-1}$ be the set of vectors in $\mathbb{R}^M$ with complement property and with additional requirement of orthogonality and normalization ($\|\cdot\| = 1$) to the sets $\{\varphi_n\}_{n=1}^{M}$ and $\{\varphi_n\}_{n=M+1}^{2M-1}$. The corollary 2 ensures the existence of such set. Let $I_k \subseteq \{1, \ldots, M\}$ for $k = 1, \ldots, M$, and $J_k \subseteq \{M+1, \ldots, 2M-1\}$ for $k = M+1, \ldots, 2M-1$, let P_{I_k} and P_{J_k} be projections on span $(\{\varphi_n\}_{n\in I_k})$ and span $(\{\varphi_n\}_{n\in J_k})$ respectively. The next construction ensures phaseless recovery for $x \in \mathbb{R}^M$ by $\left\|P_{I_k}x\right\|$ and $\left\|P_{J_k}x\right\|$ for $k = 1, \ldots, 2M-1$.

Let $A = [a_{kz}]_{k,z=1}^{M}$ be $M \times M$-matrix, its rows are agreed with I_k, i. e. $a_{kz} = 1$, for $z \in I_k$, and $a_{kz} = 0$ for other z.

Similarly we define the matrix $B = [b_{kz}]_{k,z=1}^{M-1}$ as $(M-1)\times(M-1)$-matrix with $b_{kz} = 1$ for $z + M \in J_k$, and $b_{kz} = 0$ for other z.

Let's look at the subspaces span $(\{\varphi_n\}_{n\in I_k})$ for $k = 1, \ldots, M$. For $x \in \mathbb{R}^M$ we have

$$\left\|P_{I_k}x\right\|^2 = \sum_{n\in I_k} |\langle x, \varphi_n\rangle|^2 ,$$

whence

$$\begin{bmatrix} \left\|P_{I_1}x\right\|^2 \\ \vdots \\ \left\|P_{I_M}x\right\|^2 \end{bmatrix} = A \begin{bmatrix} |\langle x, \varphi_1\rangle|^2 \\ \vdots \\ |\langle x, \varphi_M\rangle|^2 \end{bmatrix}.$$

This equation may be solved upon $\{|\langle x, \varphi_n\rangle|\}_{n=1}^{M}$, if the matrix A is invertible. Similar equation may be written with the matrix B. Hence if the matrixes A and B are invertible, we obtain the complete set of "measurements" $\{|\langle x, \varphi_n\rangle|\}_{n=1}^{2M-1}$. The set $\{\varphi_n\}_{n=1}^{2M-1}$ has complement property and according to theorem 2, phaseless recovery is possible using subspaces span $(\{\varphi_n\}_{n\in I_k})$ and span $(\{\varphi_n\}_{n\in J_k})$ for $k = 1, \ldots, 2M-1$. To complete the proof we choose $\{I_k\}_{k=1}^{M}$ and $\{J_k\}_{k=M+1}^{2M-1}$ to provide the invertibility of the matrixes A and B.

Let's note that the quantity of ones in each row coincides with the dimension of the appropriate subspace. Such selection is possible according to lemma 2 for any subsets I_k, J_k, with $1 \le |I_k| \le M-1$ and $1 \le |J_k| \le M-2$.

The answer to the next question is unknown [4]:

Question. *Is it possible phaseless recovery by norms of projections in $\mathbb{R}^M$ with the set of subspaces $\{W_n\}_{n=1}^{N}$ for $N < 2M-1$?*

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