

# Unbounded solutions for systems with hysteretic nonlinearity

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## Abstract

The work is dedicated to study of the dynamics of model systems that are described by differential equations with additive hysteretic nonlinearities in the case when a hysteretic loop bypass clockwise. In particular, for the model of the harmonic oscillator with hysteretic external force the conditions for existence of unbounded solutions are obtained. For the model of harmonic oscillator with a dry (Coulomb) and viscous friction under external hysteretic affection we obtained a set of conditions that ensure the self-oscillation mode. The numerical examples that illustrate the results of this work are also presented.

*Keywords:* unbounded solutions; hysteresis; viscous and Coulomb friction;

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## 1. Introduction

Appearance of mathematical models of hysteretic phenomena is connected with the wide set of applied problems, generally in the theory of automatic control (see, e.g., [1] and related references). In these systems the sources with hysteretic properties can not be considered separately, but should be considered as a part of some complex system [2,3]. In this way the hysteretic nonlinearity should be taken into account on the step of modeling and should be included to the system of differential equations that describe the system under consideration. Investigation of hysteretic phenomena has a long history. First works on the systems with hysteretic parts were made by A.A. Andronov in 1946. However the rigorous mathematical theory of hysteresis was formulated in the monography of M.A. Krasnoselskii and A.V. Pokrovskii [4] where the hysteretic phenomena are considered in the frame of the system's theory. Namely, the hysteretic converters are treated as operators (such operators parametrically depend on its initial state) defined on a sufficiently rich functional space (for example, in the space of continuous functions).

An important problem is the study of resonance in systems with hysteresis [5]. In [6] it is considered the dynamics of the oscillator with strong nonlinearity (authors studied its phase portrait and the trajectory). It is proved that the form of periodic solutions depends on the "origin" of strong nonlinearity. The main result of [6] is the fact that, for a class of equations that describe the harmonic oscillations with resonance external force and hysteretic operator in the right part of equation, the presence or absence of unbounded solutions depend on the amplitude of the external affection.

In this paper we study the resonance properties of systems in which the energy pumping in the system due to the presence of hysteresis level. Examples of such systems are the oscillations of the ferromagnetic ball in a magnetic field, oscillations of the system of coupled oscillators when the "connection force" has a hysteretic nature [7]. It should also be pointed out that these systems can be considered as an effective models in the solid state physics in the modeling of interatomic interactions taking into account various types of dislocations that arise due to material's aging. Also such a model can be applied for the problem of viscoelasticity.

## 2. Model

Let us consider a system whose dynamics is described by the following equation with the corresponding initial conditions:

$$\begin{aligned} \ddot{x} + \omega^2 x &= R[\omega_0]x \\ x(0) &= x_0, \quad \dot{x}(0) = x_1 \end{aligned} \tag{1}$$

where  $R$  is a non-ideal relay operator with the negative spin and  $x$  is an initial state of the operator (see Fig.1).

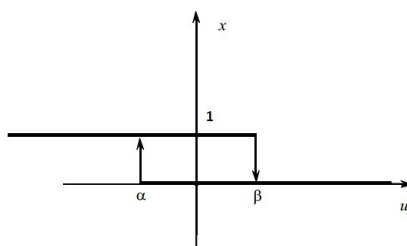
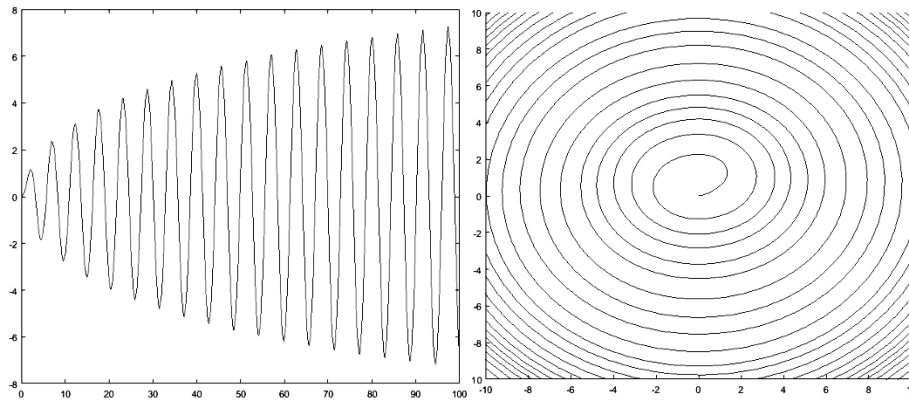


Fig. 1. Non-ideal relay with negative spin.

**Theorem:** Let the initial value satisfies the condition  $x_0 \notin [\alpha; \beta]$ . Then the corresponding solutions are unbounded. In other words, if the initial value is such that the hysteretic element “works”, then the corresponding solution is unbounded.



**Fig. 2.** Solution (left panel) and phase portrait (right panel) of equation (1) with given initial conditions.

**Note 1.** Let us note that the solution oscillates and the rate of growth of amplitude is proportional to  $\sqrt{t}$ .

**Note 2.** The theorem remains valid for other types of hysteretic nonlinearities. Main requirement is the positiveness of the loop’s area.

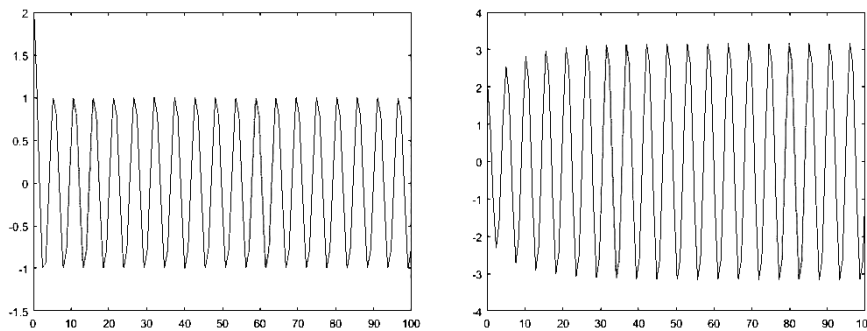
If the initial value lies on the segment  $[\alpha; \beta]$ , then the corresponding solution is limited on the whole positive time interval.

### 3. Viscous and Coulomb friction

Let us consider the various kinds of friction. The dynamics of oscillator with a viscous friction can be described by the equation:

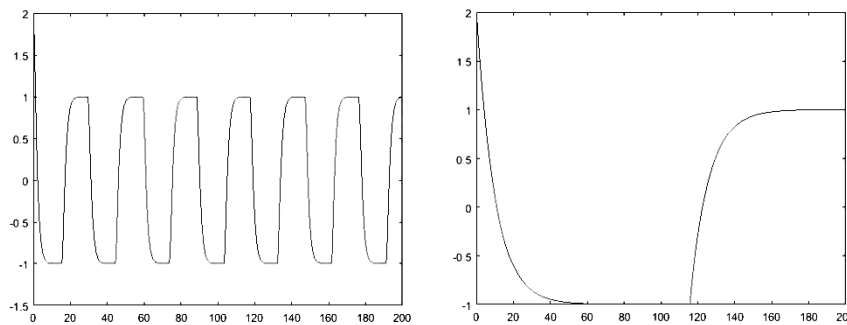
$$\ddot{x} + 2b\dot{x} + \omega^2 x = R[\omega_0]x \tag{2}$$

In the following consideration we assume that the threshold number of non-ideal relay are symmetric relative to origin. Considering the dynamics of the solution it should be noted that once the amplitude of the solution becomes high enough, the work of the friction force balances the energy obtained by the oscillator from the hysteretic element. Let us consider two cases related to the different kind of the roots of characteristic equation of the linear part of the equation (2).



**Fig. 3.** Oscillations in the system (1) at  $\xi=0.1$  and  $\xi=1$  (left and right panels, respectively).

A qualitative change in the nature of the solution becomes due to the fact that the roots of the characteristic equation are real. The nature of the oscillations has remained stable with the changed period (see Fig. 3).



**Fig. 4.** Oscillations in the system (1) at  $\xi=2.1$  and  $\xi=10$  (left and right panels, respectively).

The dynamics of oscillator with the Coulomb friction under external hysteretic affection is described by the equation

$$\ddot{x} + \eta \operatorname{sgn} \dot{x} + x = R[a, -a]x \quad (3)$$

Multiplying both sides of (3) on  $\dot{x}$  and integrating over the period  $T$  we obtain:

$$\int_0^T \frac{1}{2} \frac{d}{dt} (\dot{x}^2 + x^2) = -\eta (\operatorname{sgn} \dot{x}) \dot{x} + \dot{x} R[a, -a]x \quad (4)$$

$$\Delta E = -\eta \int_0^T |\dot{x}| dt + S_{loop} \quad (5)$$

From the equation (5) it follows that the energy gain will be positive if the work of friction forces will be smaller than the loop's area (otherwise it will be negative). In this way the considered system can be treated as a system with the negative feedback. Let us note that at steady-state regime the condition

$$2(x_{\max} - x_{\min})\eta = S_{loop} \quad (6)$$

is satisfied. This means that the amplitude of the oscillation is such that the work of frictional forces on the period is equal to the loop's area.

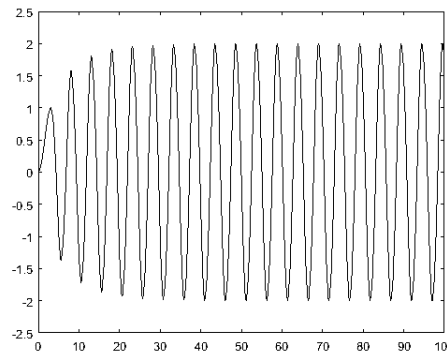


Fig. 5. Oscillations in the system (1) at  $\eta=0.5$ .

#### 4. Conclusion

In this paper we found that in the dynamical model of harmonic oscillator under external hysteretic affection (force with a hysteretic nature) with a negative spin the unbounded oscillations occur. It is shown also, that in the models of the harmonic oscillator with Coulomb and viscous frictions under hysteretic external force with negative spin the stable self-oscillations are realized.

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